

# Noether's theorem for fractional variational problems of variable order

Research Article

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**Abstract:** We prove a necessary optimality condition of Euler–Lagrange type for fractional variational problems with derivatives of incommensurate variable order. This allows us to state a version of Noether's theorem without transformation of the independent (time) variable. Considered derivatives of variable order are defined in the sense of Caputo.

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• Euler–Lagrange equations • Noether's theorem

## 1. Introduction

Fractional calculus is the scientific discipline that deals with integrals and derivatives of arbitrary real (or complex) order. Since the XVIIth century, when fractional integration and differentiation was brought up for the first time, many well-known mathematicians contributed to the theory, among them Euler, Laplace, Fourier, Abel, Liouville and Riemann. For a comprehensive knowledge of fractional calculus we refer the reader to the books [20, 21, 30, 37]. The XXth century has shown that fractional order calculus is more adequate to describe real world problems than the integer/standard calculus [9, 18]. Therefore, not only mathematicians have currently a strong interest in the fractional calculus but also researchers in applied fields such as mechanics, physics, chemistry, biology, economics, control theory and signal processing [6, 29, 39].

A generalization of the fractional calculus was proposed in 1993 by Samko and Ross [38]. They considered integrals and derivatives of order  $\alpha$ , where  $\alpha$  is not a constant but a function. Afterwards, several works were dedicated to

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these operators [35, 36]. Interesting applications of the proposed calculus are found in the theory of viscous flows and mechanics [10, 12, 23, 31, 32].

In 1996, Riewe initiated the theory of the fractional calculus of variations by considering problems of mechanics with fractional order derivatives [33, 34]. Nowadays, the fractional calculus of variations is under current strong research (see, e.g., [1, 2, 5, 7, 8, 11, 15, 16, 25, 26] and references therein). For the state of the art on the subject we refer to the recent book [24]. Here we remark that results for problems of the calculus of variations with variable order fractional operators are scarce, reducing to those found in [4, 27, 28]. In particular, no Noether's symmetry theorem of variable order has been proved before.

The main aim of the current work is to generalize the Noether theorem, originally proved by the German mathematician Emmy Noether in 1918, asserting that invariance properties of integral functionals lead to corresponding conservation laws [19, 22, 40]. In contrast with previous works [3, 13–15], where fractional versions of the Noether theorem for a constant non-integer order  $\alpha$  are obtained, here we consider more general fractional variational problems with variable order derivatives.

The paper is organized as follows. In Section 2, basic definitions and properties of variable order fractional operators are given. Sections 3, 4 and 5 contain our main results: we prove integration by parts formulas for variable order fractional integrals (Theorem 3.1) and derivatives (Theorem 3.2), a necessary optimality condition of Euler–Lagrange type for fractional variational problems with derivatives of incommensurate variable order (Theorem 4.1), and a version of Noether's theorem for such fractional variable order problems (Theorem 5.1). Then, in Section 6, we illustrate our results through an example. We finish with Section 7 of conclusion.

## 2. Variable order fractional operators

In this section we recall the basic definitions necessary in the sequel, and derive a new interesting mapping property for the fractional integral operators of variable order (Theorem 2.1). We set  $\Delta := \{(t, \tau) \in \mathbb{R}^2 : a \leq \tau < t \leq b\}$ .

**Definition 2.1 (Left and right Riemann–Liouville integrals of variable order).**

Let  $\alpha : \Delta \rightarrow (0, 1)$  and  $f \in L_1[a, b]$ . Then,

$${}_a I_t^{\alpha(\cdot, \cdot)} f(t) = \int_a^t \frac{1}{\Gamma(\alpha(t, \tau))} (t - \tau)^{\alpha(t, \tau) - 1} f(\tau) d\tau, \quad t > a,$$

is called the left Riemann–Liouville integral of variable fractional order  $\alpha(\cdot, \cdot)$ , while

$${}_t I_b^{\alpha(\cdot, \cdot)} f(t) = \int_t^b \frac{1}{\Gamma(\alpha(\tau, t))} (\tau - t)^{\alpha(\tau, t) - 1} f(\tau) d\tau, \quad t < b,$$

denotes the right Riemann–Liouville integral of variable fractional order  $\alpha(\cdot, \cdot)$ .

**Theorem 2.1.**

Let  $\alpha(t, \tau) = \alpha(t - \tau)$  with  $0 < \alpha(t - \tau) < 1 - \frac{1}{n}$  for all  $0 < t - \tau \leq b$  and a certain  $n \in \mathbb{N}$  greater or equal than two. If  $f \in AC[0, b]$ , then

$${}_0I_t^{1-\alpha(\cdot, \cdot)} f(t) = \int_0^t \frac{1}{\Gamma(1-\alpha(t-\tau))} (t-\tau)^{-\alpha(t-\tau)} f(\tau) d\tau \in AC[0, b].$$

If  $f \in AC[-b, 0]$ , then

$${}_tI_0^{1-\alpha(\cdot, \cdot)} f(t) = \int_t^0 \frac{1}{\Gamma(1-\alpha(\tau-t))} (\tau-t)^{-\alpha(\tau-t)} f(\tau) d\tau \in AC[-b, 0].$$

*Proof.* We give the proof for the left integral; the other case being proved similarly. Let  $k(s) := \frac{1}{\Gamma(1-\alpha(s))} s^{-\alpha(s)}$ . Since  $0 < \alpha(s) < 1 - \frac{1}{n}$ ,

1. for  $s \leq 1$  we have  $\ln s \geq 0$  and  $s^{-\alpha(s)} < 1$ ,
2. for  $s < 1$  we have  $\ln s < 0$  and  $s^{-\alpha(s)} < s^{\frac{1}{n}-1}$ .

Therefore,

$$\int_0^b |k(s)| d\tau = \int_0^b \left| \frac{1}{\Gamma(1-\alpha(s))} s^{-\alpha(s)} \right| d\tau < \int_0^1 \frac{1}{\Gamma(1-\alpha(s))} ds + \int_1^b \frac{1}{\Gamma(1-\alpha(s))} s^{\frac{1}{n}-1} ds.$$

Using the inequality

$$\Gamma(x+1) \geq \frac{x^2+1}{x+1}, \quad (1)$$

valid for all  $x \in [0, 1]$  (see [17]), we obtain that

$$\int_0^1 \frac{1}{\Gamma(1-\alpha(s))} ds + \int_1^b \frac{1}{\Gamma(1-\alpha(s))} s^{\frac{1}{n}-1} ds < \int_0^1 ds + \int_1^b s^{\frac{1}{n}-1} ds = 1 + nb^{\frac{1}{n}} - n < \infty.$$

It means that  $k(s) \in L_1[0, b]$ . Moreover, the condition  $f \in AC([0, b])$  implies that

$$f(t) = \int_0^t g(\tau) d\tau + f(0), \text{ where } g \in L_1([0, b]).$$

Define

$$h(t) := \int_0^t \frac{1}{\Gamma(1-\alpha(s))} s^{-\alpha(s)} g(t-s) ds$$

and integrate:

$$\int_0^t h(\theta) d\theta = \int_0^t d\theta \int_0^\theta \frac{1}{\Gamma(1-\alpha(s))} s^{-\alpha(s)} g(\theta-s) ds. \quad (2)$$

Observe that

$$\begin{aligned} \int_0^t \left( \int_s^t \left| \frac{1}{\Gamma(1-\alpha(s))} s^{-\alpha(s)} g(\theta-s) \right| d\theta \right) ds &= \int_0^t \left( \left| \frac{1}{\Gamma(1-\alpha(s))} s^{-\alpha(s)} \right| \left( \int_s^t |g(\theta-s)| d\theta \right) \right) ds \\ &< \|g\|_1 \int_0^b \left| \frac{1}{\Gamma(1-\alpha(s))} s^{-\alpha(s)} \right| ds \\ &< \infty. \end{aligned}$$

It means that assumptions of Fubini's theorem are satisfied, so we can change an order of integration in (2).

Hence,

$$\int_0^t h(\theta) d\theta = \int_0^t ds \int_s^t \frac{1}{\Gamma(1-\alpha(s))} s^{-\alpha(s)} g(\theta-s) d\theta = \int_0^t \frac{1}{\Gamma(1-\alpha(s))} s^{-\alpha(s)} ds \int_s^t g(\theta-s) d\theta.$$

Putting  $\xi = \theta - s$ , one has  $d\xi = d\theta$  and we get

$$\int_0^t h(\theta) d\theta = \int_0^t \frac{1}{\Gamma(1-\alpha(s))} s^{-\alpha(s)} ds \int_0^{t-s} g(\xi) d\xi.$$

But  $\int_0^{t-s} g(\xi) d\xi = f(t-s) - f(0)$  and, therefore, the following equality holds:

$$\int_0^t h(\theta) d\theta = \int_0^t \frac{1}{\Gamma(1-\alpha(s))} s^{-\alpha(s)} f(t-s) ds - f(0) \int_0^t \frac{1}{\Gamma(1-\alpha(s))} s^{-\alpha(s)} ds.$$

Putting  $\tau = t - s$ ,  $d\tau = -ds$  and

$$\int_0^t h(\theta) d\theta = \int_0^t \frac{1}{\Gamma(1-\alpha(t-\tau))} (t-\tau)^{-\alpha(t-\tau)} f(\tau) d\tau - f(0) \int_0^t \frac{1}{\Gamma(1-\alpha(t-\tau))} (t-\tau)^{-\alpha(t-\tau)} d\tau.$$

Both functions on the right-hand side of the equality

$$\int_0^t \frac{1}{\Gamma(1-\alpha(t-\tau))} (t-\tau)^{-\alpha(t-\tau)} f(\tau) d\tau = \int_0^t h(\theta) d\theta + f(0) \int_0^t \frac{1}{\Gamma(1-\alpha(t-\tau))} (t-\tau)^{-\alpha(t-\tau)} d\tau$$

belong to  $AC([0, b])$ , hence  ${}_0I_t^{\alpha(\cdot, \cdot)} f \in AC[0, b]$ . □

**Definition 2.2 (Left and right Riemann–Liouville derivatives of variable order).**

Let  $\alpha : \Delta \rightarrow (0, 1)$ . If  ${}_aI_t^{1-\alpha(\cdot, \cdot)} f \in AC[a, b]$ , then the left Riemann–Liouville derivative of variable fractional order  $\alpha(\cdot, \cdot)$  is defined by

$${}_aD_t^{\alpha(\cdot, \cdot)} f(t) = \frac{d}{dt} {}_aI_t^{1-\alpha(\cdot, \cdot)} f(t) = \frac{d}{dt} \int_a^t \frac{1}{\Gamma(1-\alpha(t, \tau))} (t-\tau)^{-\alpha(t, \tau)} f(\tau) d\tau, \quad t > a,$$

while the right Riemann–Liouville derivative of variable order  $\alpha(\cdot, \cdot)$  is defined for functions  $f$  such that  ${}_tI_b^{1-\alpha(\cdot, \cdot)} f \in AC[a, b]$  by

$${}_tD_b^{\alpha(\cdot, \cdot)} f(t) = -\frac{d}{dt} {}_tI_b^{1-\alpha(\cdot, \cdot)} f(t) = \frac{d}{dt} \int_t^b \frac{-1}{\Gamma(1-\alpha(\tau, t))} (\tau-t)^{-\alpha(\tau, t)} f(\tau) d\tau, \quad t < b.$$

**Definition 2.3 (Left and right Caputo derivatives of variable fractional order).**

Let  $0 < \alpha(t, \tau) < 1$  for all  $t, \tau \in [a, b]$ . If  $f \in AC[a, b]$ , then the left Caputo derivative of variable fractional order  $\alpha(\cdot, \cdot)$  is defined by

$${}_a^C D_t^{\alpha(\cdot, \cdot)} f(t) = \int_a^t \frac{1}{\Gamma(1 - \alpha(t, \tau))} (t - \tau)^{-\alpha(t, \tau)} \frac{d}{d\tau} f(\tau) d\tau, \quad t > a,$$

while the right Caputo derivative of variable fractional order  $\alpha(\cdot, \cdot)$  is given by

$${}_t^C D_b^{\alpha(\cdot, \cdot)} f(t) = \int_t^b \frac{-1}{\Gamma(1 - \alpha(\tau, t))} (\tau - t)^{-\alpha(\tau, t)} \frac{d}{d\tau} f(\tau) d\tau, \quad t < b.$$

### 3. Variable order fractional integration by parts

In this section we derive integration by parts formulas, which are essential for proving Euler–Lagrange equations.

**Theorem 3.1 (Integration by parts for variable order fractional integrals).**

If  $\frac{1}{n} < \alpha(t, \tau) < 1$  for all  $(t, \tau) \in \Delta$  and a certain  $n \in \mathbb{N}$  greater or equal than two, and  $f, g \in C([a, b]; \mathbb{R})$ , then

$$\int_a^b g(t) {}_a I_t^{\alpha(\cdot, \cdot)} f(t) dt = \int_a^b f(t) {}_t I_b^{\alpha(\cdot, \cdot)} g(t) dt.$$

*Proof.* Define

$$F(\tau, t) := \begin{cases} \left| \frac{1}{\Gamma(\alpha(t, \tau))} (t - \tau)^{\alpha(t, \tau)-1} g(t) f(\tau) \right| & \text{if } \tau < t, \\ 0 & \text{if } \tau \geq t, \end{cases}$$

for all  $(\tau, t) \in [a, b] \times [a, b]$ . Since  $f$  and  $g$  are continuous functions on  $[a, b]$ , they are bounded on  $[a, b]$ , i.e., there exist  $C_1, C_2 > 0$  such that  $|g(t)| \leq C_1$  and  $|f(t)| \leq C_2$ ,  $t \in [a, b]$ . Therefore,

$$\begin{aligned} \int_a^b \left( \int_a^b F(\tau, t) d\tau \right) dt &= \int_a^b \left( \int_a^t \left| \frac{1}{\Gamma(\alpha(t, \tau))} (t - \tau)^{\alpha(t, \tau)-1} g(t) f(\tau) \right| d\tau \right) dt \\ &\leq C_1 C_2 \int_a^b \left( \int_a^t \left| \frac{1}{\Gamma(\alpha(t, \tau))} (t - \tau)^{\alpha(t, \tau)-1} \right| d\tau \right) dt \\ &= C_1 C_2 \int_a^b \left( \int_a^t \frac{1}{\Gamma(\alpha(t, \tau))} (t - \tau)^{\alpha(t, \tau)-1} d\tau \right) dt. \end{aligned}$$

Because  $\frac{1}{n} < \alpha(t, \tau) < 1$ ,

1. for  $1 \leq t - \tau$  we have  $\ln(t - \tau) \geq 0$  and  $(t - \tau)^{\alpha(t, \tau)-1} < 1$ ;
2. for  $1 > t - \tau$  we have  $\ln(t - \tau) < 0$  and  $(t - \tau)^{\alpha(t, \tau)-1} < (t - \tau)^{\frac{1}{n}-1}$ .

Therefore,

$$\begin{aligned} C_1 C_2 \int_a^b \left( \int_a^t \frac{1}{\Gamma(\alpha(t, \tau))} (t - \tau)^{\alpha(t, \tau)-1} d\tau \right) dt \\ < C_1 C_2 \int_a^b \left( \int_a^{t-1} \frac{1}{\Gamma(\alpha(t, \tau))} d\tau + \int_{t-1}^t \frac{1}{\Gamma(\alpha(t, \tau))} (t - \tau)^{\frac{1}{n}-1} d\tau \right) dt. \end{aligned}$$

Moreover, by inequality (1), valid for  $x \in [0, 1]$ , one has

$$\begin{aligned}
 & C_1 C_2 \int_a^b \left( \int_a^{t-1} \frac{1}{\Gamma(\alpha(t, \tau))} d\tau + \int_{t-1}^t \frac{1}{\Gamma(\alpha(t, \tau))} (t - \tau)^{\frac{1}{n}-1} d\tau \right) dt \\
 & \leq C_1 C_2 \int_a^b \left( \int_a^{t-1} \frac{\alpha^2(t, \tau) + \alpha(t, \tau)}{\alpha^2(t, \tau) + 1} d\tau + \int_{t-1}^t \frac{\alpha^2(t, \tau) + \alpha(t, \tau)}{\alpha^2(t, \tau) + 1} (t - \tau)^{\frac{1}{n}-1} d\tau \right) dt \\
 & < C_1 C_2 \int_a^b \left( \int_a^{t-1} d\tau + \int_{t-1}^t (t - \tau)^{\frac{1}{n}-1} d\tau \right) dt \\
 & = C_1 C_2 (b - a) \left( \frac{b + a}{2} - 1 + n - a \right) \\
 & < \infty.
 \end{aligned}$$

Hence, one can use the Fubini theorem to change the order of integration:

$$\begin{aligned}
 \int_a^b g(t) {}_a I_t^{\alpha(\cdot, \cdot)} f(t) dt &= \int_a^b \left( \int_a^t g(t) f(\tau) \frac{1}{\Gamma(\alpha(t, \tau))} (t - \tau)^{\alpha(t, \tau)-1} d\tau \right) dt \\
 &= \int_a^b \left( \int_\tau^b g(t) f(\tau) \frac{1}{\Gamma(\alpha(t, \tau))} (t - \tau)^{\alpha(t, \tau)-1} dt \right) d\tau \\
 &= \int_a^b f(\tau) {}_\tau I_b^{\alpha(\cdot, \cdot)} g(\tau) d\tau.
 \end{aligned}$$

□

**Theorem 3.2 (Integration by parts for variable order fractional derivatives).**

Let  $0 < \alpha(t, \tau) < 1 - \frac{1}{n}$  for all  $(t, \tau) \in \Delta$  and a certain  $n \in \mathbb{N}$  greater or equal than two. If  $f \in C^1([a, b]; \mathbb{R})$ ,  $g \in C([a, b]; \mathbb{R})$ , and  ${}_t I_b^{1-\alpha(\cdot, \cdot)} g$ , then

$$\int_a^b g(t) {}_a^C D_t^{\alpha(\cdot, \cdot)} f(t) dt = f(t) {}_t I_b^{1-\alpha(\cdot, \cdot)} g(t) \Big|_a^b + \int_a^b f(t) {}_t D_b^{\alpha(\cdot, \cdot)} g(t) dt.$$

If  $f \in C^1([a, b]; \mathbb{R})$ ,  $g \in C([a, b]; \mathbb{R})$ , and  ${}_a I_t^{1-\alpha(\cdot, \cdot)} g \in AC[a, b]$ , then

$$\int_a^b g(t) {}_t^C D_b^{\alpha(\cdot, \cdot)} f(t) dt = -f(t) {}_a I_t^{1-\alpha(\cdot, \cdot)} g(t) \Big|_a^b + \int_a^b f(t) {}_a D_t^{\alpha(\cdot, \cdot)} g(t) dt.$$

*Proof.* By Definition 2.3, it follows that  ${}_a^C D_t^{\alpha(\cdot, \cdot)} f(t) = {}_a I_t^{1-\alpha(\cdot, \cdot)} \frac{d}{dt} f(t)$ . Applying Theorem 3.1 and integration by parts for classical (integer order) derivatives, we obtain

$$\begin{aligned}
 \int_a^b g(t) {}_a^C D_t^{\alpha(\cdot, \cdot)} f(t) dt &= \int_a^b g(t) {}_a I_t^{1-\alpha(\cdot, \cdot)} \frac{d}{dt} f(t) dt = \int_a^b \frac{d}{dt} f(t) {}_t I_b^{1-\alpha(\cdot, \cdot)} g(t) dt \\
 &= f(t) {}_t I_b^{1-\alpha(\cdot, \cdot)} g(t) \Big|_a^b - \int_a^b f(t) \frac{d}{dt} {}_t I_b^{1-\alpha(\cdot, \cdot)} g(t) dt \\
 &= f(t) {}_t I_b^{1-\alpha(\cdot, \cdot)} g(t) \Big|_a^b + \int_a^b f(t) {}_t D_b^{\alpha(\cdot, \cdot)} g(t) dt.
 \end{aligned}$$

The second formula is proved in a similar way. □

## 4. Euler–Lagrange equations for incommensurate fractional variational problems of variable order

Now we prove necessary optimality conditions of Euler–Lagrange type for incommensurate order fractional variational problems. Let  $\alpha_i(t, \tau)$ ,  $\beta_i(t, \tau)$ ,  $i = 1, \dots, n$ ,  $n \in \mathbb{N}$ , satisfy the assumptions of Theorem 3.2. Consider the following problem.

### Problem 4.1.

Find a function  $q = q(t)$  for which the functional

$$\mathcal{J}[q] = \int_a^b L \left( t, q(t), {}^C D_t^{\alpha_1(t, \tau)} q(t), \dots, {}^C D_t^{\alpha_n(t, \tau)} q(t), {}^C D_b^{\beta_1(t, \tau)} q(t), \dots, {}^C D_b^{\beta_n(t, \tau)} q(t) \right) dt \quad (3)$$

attains an extremum on the set

$$\mathcal{D} = \left\{ q \in C^1([a, b]; \mathbb{R}) : q(a) = q_a, q(b) = q_b \text{ and } {}^C D_t^{\alpha_i(t, \tau)} q, {}^C D_b^{\beta_i(t, \tau)} q \in C([a, b]; \mathbb{R}), i = 1, \dots, n \right\}.$$

We assume that  $L \in C^1([a, b] \times \mathbb{R}^{2n+1}; \mathbb{R})$ ;  $t \mapsto \partial_{i+2} L$  is continuous, has absolutely continuous integral  ${}_t I_b^{1-\alpha_i(t, \tau)}$  and continuous derivative  ${}_t D_b^{\alpha_i(t, \tau)}$  for each  $i = 1, \dots, n$ ;  $t \mapsto \partial_{n+i+2} L$  is continuous, has absolutely continuous integral  ${}_a I_t^{1-\beta_i(t, \tau)}$  and continuous derivative  ${}_a D_t^{\beta_i(t, \tau)}$  for each  $i = 1, \dots, n$ . For simplicity of notation, we introduce the following notation:

$$\{q, \alpha, \beta\}(t) := \left( t, q(t), {}^C D_t^{\alpha(t, \tau)} q(t), {}^C D_b^{\beta(t, \tau)} q(t) \right),$$

where

$${}_a^C D_t^{\alpha(t, \tau)} := \left( {}^C D_t^{\alpha_1(t, \tau)}, \dots, {}^C D_t^{\alpha_n(t, \tau)} \right), \quad {}^C D_b^{\beta(t, \tau)} := \left( {}^C D_b^{\beta_1(t, \tau)}, \dots, {}^C D_b^{\beta_n(t, \tau)} \right).$$

### Definition 4.1.

Let  $f \in AC([a, b])$ . For  ${}_t I_b^{1-\gamma(t, \tau)} g \in AC([a, b])$  we define the following operator:

$$D_-^{\gamma(t, \tau)}[f, g] := -f {}_t D_b^{\gamma(t, \tau)} g + g {}_a^C D_t^{\gamma(t, \tau)}[f], \quad (4)$$

while for  ${}_a I_t^{1-\gamma(t, \tau)} g \in AC([a, b])$  we define

$$D_+^{\gamma(t, \tau)}[f, g] := -f {}_a D_t^{\gamma(t, \tau)}[g] + g {}_t^C D_b^{\gamma(t, \tau)}[f]. \quad (5)$$

### Remark 4.1.

If  $\gamma(t, \tau) \equiv \gamma$  is a constant function, then

$$\lim_{\gamma \rightarrow 1^-} D_-^\gamma[f, g] = fg' + gf' = \frac{d}{dt}(fg) = \lim_{\gamma \rightarrow 1^-} D_-^\gamma[g, f]$$

and

$$\lim_{\gamma \rightarrow 1^-} D_+^\gamma[f, g] = -fg' - gf' = -\frac{d}{dt}(fg) = \lim_{\gamma \rightarrow 1^-} D_+^\gamma[g, f].$$

**Theorem 4.1.**

Let function  $q$  be a solution to Problem 4.1. Then,

$$\partial_2 L \{q, \alpha, \beta\}(t) - \sum_{i=1}^n D_-^{\alpha_i(t, \tau)} [1, \partial_{i+2} L \{q, \alpha, \beta\}(t)] - \sum_{i=1}^n D_+^{\beta_i(t, \tau)} [1, \partial_{n+2+i} L \{q, \alpha, \beta\}(t)] = 0. \quad (6)$$

*Proof.* Assume that  $q$  is an extremizer of  $\mathcal{J}$ . Consider the value of  $\mathcal{J}$  at nearby function  $\hat{q}(t) = q(t) + \varepsilon \eta(t)$ , where  $\varepsilon \in \mathbb{R}$  is a small parameter and  $\eta \in C^1([a, b]; \mathbb{R})$  is an arbitrary function satisfying  $\eta(a) = \eta(b) = 0$  and such that  ${}_a^C D_t^{\alpha_i(t, \tau)} \hat{\eta}$  and  ${}_t^C D_b^{\beta_i(t, \tau)} \hat{\eta}$  are continuous. Let

$$J(\varepsilon) = \mathcal{J}[\hat{q}] = \int_a^b L \{\hat{q}, \alpha, \beta\}(t) dt.$$

A necessary condition for  $\hat{q}$  to be an extremizer is given by

$$\begin{aligned} \left. \frac{dJ}{d\varepsilon} \right|_{\varepsilon=0} = 0 \Leftrightarrow \int_a^b \left( \partial_2 L \{q, \alpha, \beta\}(t) \eta(t) + \sum_{i=1}^n \partial_{i+2} L \{q, \alpha, \beta\}(t) {}_a^C D_t^{\alpha_i(t, \tau)} \eta(t) \right. \\ \left. + \sum_{i=1}^n \partial_{n+2+i} L \{q, \alpha, \beta\}(t) {}_t^C D_b^{\beta_i(t, \tau)} \eta(t) \right) dt = 0. \end{aligned} \quad (7)$$

Using the variable order fractional integration by parts formulas (Theorem 3.2), we obtain that

$$\begin{aligned} \int_a^b \partial_{i+2} L \{q, \alpha, \beta\}(t) {}_a^C D_t^{\alpha_i(t, \tau)} \eta(t) dt \\ = \eta(t) {}_t I_b^{1-\alpha_i(t, \tau)} \partial_{i+2} L \{q, \alpha, \beta\}(t) \Big|_a^b + \int_a^b \eta(t) {}_t D_b^{\alpha_i(t, \tau)} \partial_{i+2} L \{q, \alpha, \beta\}(t) dt, \quad i = 1, \dots, n, \end{aligned}$$

and

$$\begin{aligned} \int_a^b \partial_{n+i+2} L \{q, \alpha, \beta\}(t) {}_t^C D_b^{\beta_i(t, \tau)} \eta(t) dt \\ = - \eta(t) {}_a I_t^{1-\beta_i(t, \tau)} \partial_{n+i+2} L \{q, \alpha, \beta\}(t) \Big|_a^b + \int_a^b \eta(t) {}_a D_t^{\beta_i(t, \tau)} \partial_{n+i+2} L \{q, \alpha, \beta\}(t) dt, \quad i = 1, \dots, n. \end{aligned}$$

Because  $\eta(a) = \eta(b) = 0$ , (7) simplifies to

$$\int_a^b \eta(t) \left( \partial_2 L \{q, \alpha, \beta\}(t) + \sum_{i=1}^n {}_t D_b^{\alpha_i(t, \tau)} \partial_{i+2} L \{q, \alpha, \beta\}(t) + \sum_{i=1}^n {}_a D_t^{\beta_i(t, \tau)} \partial_{n+i+2} L \{q, \alpha, \beta\}(t) \right) dt = 0.$$

Applying the fundamental lemma of the calculus of variations (see, e.g., [40]), we obtain that

$$\partial_2 L \{q, \alpha, \beta\}(t) + \sum_{i=1}^n {}_t D_b^{\alpha_i(t, \tau)} \partial_{i+2} L \{q, \alpha, \beta\}(t) + \sum_{i=1}^n {}_a D_t^{\beta_i(t, \tau)} \partial_{n+i+2} L \{q, \alpha, \beta\}(t) = 0.$$

Finally, by Definition 4.1, we arrive to (6). □



## 5. Noether's theorem of fractional variable order

Now we show, employing operators (4) and (5), that invariance of (3) conducts to a variable order version of a fractional conservation law. To prove this, we borrow the method from [13, 15, 19], where Noether's theorem is stated for fractional derivatives of constant order.

### Definition 5.1.

A function  $q$  that is a solution to (6) is said to be a variable order fractional extremal for functional  $\mathcal{J}$ .

### Definition 5.2.

We say that functional (3) is invariant under an  $\varepsilon$ -parameter group of infinitesimal transformations

$$\bar{q}(t) = q(t) + \varepsilon \xi(t, q(t)) + o(\varepsilon) \quad (8)$$

if

$$\int_{t_a}^{t_b} L \left( t, q(t), {}^C D_t^{\alpha(t,\tau)} q(t), {}^C D_b^{\beta(t,\tau)} q(t) \right) dt = \int_{t_a}^{t_b} L \left( t, \bar{q}(t), {}^C D_t^{\alpha(t,\tau)} \bar{q}(t), {}^C D_b^{\beta(t,\tau)} \bar{q}(t) \right) dt \quad (9)$$

for any subinterval  $[t_a, t_b] \subseteq [a, b]$ .

### Lemma 5.1 (Necessary condition of invariance).

If functional (3) is invariant under an  $\varepsilon$ -parameter group of infinitesimal transformations (8), then

$$\begin{aligned} \partial_2 L \{q, \alpha, \beta\} (t) \xi(t, q(t)) + \sum_{i=1}^n \partial_{i+2} L \{q, \alpha, \beta\} (t) {}^C D_t^{\alpha_i(t,\tau)} \xi(t, q(t)) \\ + \sum_{i=1}^n \partial_{n+i+2} L \{q, \alpha, \beta\} (t) {}^C D_b^{\beta_i(t,\tau)} \xi(t, q(t)) = 0. \end{aligned} \quad (10)$$

*Proof.* Since, by hypothesis, condition (9) is satisfied for any subinterval  $[t_a, t_b] \subseteq [a, b]$ , we have

$$L \left( t, q(t), {}^C D_t^{\alpha(t,\tau)} q(t), {}^C D_b^{\beta(t,\tau)} q(t) \right) = L \left( t, \bar{q}(t), {}^C D_t^{\alpha(t,\tau)} \bar{q}(t), {}^C D_b^{\beta(t,\tau)} \bar{q}(t) \right). \quad (11)$$

Now, differentiating (11) with respect to  $\varepsilon$ , then putting  $\varepsilon = 0$ , and applying definitions and properties of variable order Caputo fractional derivatives, we obtain that

$$\begin{aligned} 0 &= \partial_2 L \{q, \alpha, \beta\} (t) \xi(t, q(t)) \\ &+ \sum_{i=1}^n \partial_{i+2} L \{q, \alpha, \beta\} (t) \frac{d}{d\varepsilon} \left[ \int_a^t \frac{1}{\Gamma(1 - \alpha_i(t, \tau))} (t - \tau)^{-\alpha_i(t, \tau)} \frac{d}{d\tau} \bar{q}(\tau) d\tau \right]_{\varepsilon=0} \\ &+ \sum_{i=1}^n \partial_{n+i+2} L \{q, \alpha, \beta\} (t) \frac{d}{d\varepsilon} \left[ \int_t^b \frac{-1}{\Gamma(1 - \beta_i(t, \tau))} (\tau - t)^{-\beta_i(t, \tau)} \frac{d}{d\tau} \bar{q}(\tau) d\tau \right]_{\varepsilon=0} \\ &= \partial_2 L \{q, \alpha, \beta\} (t) \xi(t, q(t)) + \sum_{i=1}^n \partial_{i+2} L \{q, \alpha, \beta\} (t) {}^C D_t^{\alpha_i(t,\tau)} \xi(t, q(t)) \\ &+ \sum_{i=1}^n \partial_{n+i+2} L \{q, \alpha, \beta\} (t) {}^C D_b^{\beta_i(t,\tau)} \xi(t, q(t)). \end{aligned}$$

□

**Theorem 5.1 (Noether's theorem for variable order fractional variational problems).**

If functional (3) is invariant in the sense of Definition 5.2, then

$$\sum_{i=1}^n D_-^{\alpha_i(t,\tau)} [\xi(t, q(t)), \partial_{i+2} L \{q, \alpha, \beta\}(t)] + \sum_{i=1}^n D_+^{\beta_i(t,\tau)} [\xi(t, q(t)), \partial_{n+i+2} L \{q, \alpha, \beta\}(t)] = 0, \quad t \in [a, b],$$

along all variable order fractional extremals  $q(\cdot)$ .

*Proof.* By Theorem 4.1 we have

$$\partial_2 L \{q, \alpha, \beta\}(t) = - \sum_{i=1}^n {}_t D_b^{\alpha_i(t,\tau)} \partial_{i+2} L \{q, \alpha, \beta\}(t) - \sum_{i=1}^n {}_a D_t^{\beta_i(t,\tau)} \partial_{n+i+2} L \{q, \alpha, \beta\}(t). \quad (12)$$

Substituting (12) into (10), we obtain

$$\begin{aligned} & - \sum_{i=1}^n \xi(t, q(t)) {}_t D_b^{\alpha_i(t,\tau)} \partial_{i+2} L \{q, \alpha, \beta\}(t) - \sum_{i=1}^n \xi(t, q(t)) {}_a D_t^{\beta_i(t,\tau)} \partial_{n+i+2} L \{q, \alpha, \beta\}(t) \\ & + \sum_{i=1}^n \partial_{i+2} L \{q, \alpha, \beta\}(t) {}_a^C D_t^{\alpha_i(t,\tau)} \xi(t, q(t)) + \sum_{i=1}^n \partial_{n+i+2} L \{q, \alpha, \beta\}(t) {}_t^C D_b^{\beta_i(t,\tau)} \xi(t, q(t)) = 0. \end{aligned}$$

Finally, by Definition 4.1, one has

$$\sum_{i=1}^n D_-^{\alpha_i(t,\tau)} [\xi(t, q(t)), \partial_{i+2} L \{q, \alpha, \beta\}(t)] + \sum_{i=1}^n D_+^{\beta_i(t,\tau)} [\xi(t, q(t)), \partial_{n+i+2} L \{q, \alpha, \beta\}(t)] = 0.$$

□

## 6. An illustrative example

Let  $\alpha$  and  $\beta$  be two functions such that  $0 < \alpha(t, \tau), \beta(t, \tau) < 1 - \frac{1}{l}$  for a certain natural number  $l$  greater or equal than two. Consider the following problem:

$$\begin{aligned} \mathcal{J}[q] &= \int_a^b L \left( t, {}_a^C D_t^{\alpha(t,\tau)} q(t), {}_t^C D_b^{\beta(t,\tau)} q(t) \right) dt \longrightarrow \text{extremize} \\ q(a) &= q_a, \quad q(b) = q_b. \end{aligned} \quad (13)$$

For the transformation

$$\bar{q}(t) = q(t) + \varepsilon c + o(\varepsilon), \quad (14)$$

where  $c$  is a constant, we have

$$\int_{t_a}^{t_b} L \left( t, {}_a^C D_t^{\alpha(t,\tau)} q(t), {}_t^C D_b^{\beta(t,\tau)} q(t) \right) dt = \int_{t_a}^{t_b} L \left( t, {}_a^C D_t^{\alpha(t,\tau)} \bar{q}(t), {}_t^C D_b^{\beta(t,\tau)} \bar{q}(t) \right) dt$$

for any  $[t_a, t_b] \subseteq [a, b]$ . Therefore,  $\mathcal{J}[q]$  is invariant under (14) and Noether's theorem (Theorem 5.1) asserts that

$$D_-^{\alpha(t,\tau)} \left[ c, \partial_2 L \left( t, {}_a^C D_t^{\alpha(t,\tau)} q(t), {}_t^C D_b^{\beta(t,\tau)} q(t) \right) \right] + D_+^{\beta(t,\tau)} \left[ c, \partial_3 L \left( t, {}_a^C D_t^{\alpha(t,\tau)} q(t), {}_t^C D_b^{\beta(t,\tau)} q(t) \right) \right] = 0$$

along any extremal  $q(t)$  of (13).

## 7. Conclusion

Noether's symmetry theorem, establishing that variational invariance imply conservation laws, is one of the most important results in physics and the calculus of variations. Such conservation laws are, however, only valid for conservative systems. Nonconservative forces, like friction, remove energy from the systems and, as a consequence, Noether's conservation laws cease to be valid. In order to cope with dissipative forces that do not store energy, one possibility is to use fractional calculus [33, 34]. The study of problems of the calculus of variations with fractional derivatives of variable order is, however, a rather recent subject, and available results reduce to those available at [4, 27, 28]. Here we obtain Euler–Lagrange optimality conditions for variational problems with fractional derivatives of incommensurate variable order and, for such fractional Euler–Lagrange extremals, we provide a notion of conservation law and prove a version of Noether's theorem.

The variable order calculus of variations is underdeveloped and much remains to be done. We mention here one important question that remains open. The Noether type theorem we obtain here only includes the terms related to “conservation of momentum”. It would be important to obtain a more general form of the Noether theorem with “energy” terms.

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## References

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- [1] O. P. Agrawal, S. I. Muslih, D. Baleanu, *Generalized variational calculus in terms of multi-parameters fractional derivatives*. Commun. Nonlinear Sci. Numer. Simul. **16** (2011), 4756–4767.
  - [2] R. Almeida, D. F. M. Torres, *Calculus of variations with fractional derivatives and fractional integrals*. Appl. Math. Lett. **22** (2009), 1816–1820. [arXiv:0907.1024](#)
  - [3] T. M. Atanacković, S. Konjik, S. Pilipović, S. Simić, *Variational problems with fractional derivatives: invari-*

- ance conditions and Noether's theorem. *Nonlinear Anal.* **71** (2009), 1504–1517.
- [4] T. M. Atanackovic, S. Pilipovic, *Hamilton's principle with variable order fractional derivatives*. *Fract. Calc. Appl. Anal.* **14** (2011), 94–109.
  - [5] D. Baleanu, T. Avkar, *Lagrangians with linear velocities within Riemann-Liouville fractional derivatives*. *Nuovo Cimento Soc. Ital. Fis. B* **119** (2004), 73–79. [arXiv:math-ph/0405012](#)
  - [6] D. Baleanu, K. Diethelm, E. Scalas, J. J. Trujillo, *Fractional calculus*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012.
  - [7] D. Baleanu, S. I. Muslih, *Lagrangian formulation of classical fields within Riemann-Liouville fractional derivatives*. *Phys. Scripta* **72** (2005), 119–121.
  - [8] N. R. O. Bastos, R. A. C. Ferreira, D. F. M. Torres, *Necessary optimality conditions for fractional difference problems of the calculus of variations*. *Discrete Contin. Dyn. Syst.* **29** (2011), 417–437. [arXiv:1007.0594](#)
  - [9] R. Caponetto, G. Dongola, L. Fortuna, I. Petras, *Fractional Order Systems: Modeling and Control Applications*. World Scientific Company, 2010.
  - [10] C. F. M. Coimbra, *Mechanics with variable-order differential operators*. *Ann. Phys.* **12** (2003), 692–703.
  - [11] J. Cresson, *Fractional embedding of differential operators and Lagrangian systems*. *J. Math. Phys.* **48** (2007), 033504, 34 pp. [arXiv:math/0605752](#)
  - [12] G. Diaz, C. F. M. Coimbra, *Nonlinear dynamics and control of a variable order oscillator with application to the van der Pol equation*. *Nonlinear Dynam.* **56** (2009), 145–157.
  - [13] G. S. F. Frederico, D. F. M. Torres, *A formulation of Noether's theorem for fractional problems of calculus of variations*. *J. Math. Anal. Appl.* **334** (2007), 834–846. [arXiv:math/0701187](#)
  - [14] G. S. F. Frederico, D. F. M. Torres, *Fractional conservation laws in optimal control theory*. *Nonlinear Dynam.* **53** (2008), 215–222. [arXiv:0711.0609](#)
  - [15] G. S. F. Frederico, D. F. M. Torres, *Fractional Noether's theorem in the Riesz-Caputo sense*. *Appl. Math. Comput.* **217** (2010), 1023–1033. [arXiv:1001.4507](#)
  - [16] M. A. E. Herzallah, D. Baleanu, *Fractional Euler-Lagrange equations revisited*. *Nonlinear Dynam.* **69** (2012), 977–982.
  - [17] P. Ivady, *A note on a gamma function inequality*. *J. Math. Inequal.* **3** (2009), 227–236.
  - [18] F. Jarad, T. Abdeljawad, D. Baleanu, *Stability of  $q$ -fractional non-autonomous systems*. *Nonlinear Anal. Real World Appl.* **14** (2013), 780–784.
  - [19] J. Jost, X. Li-Jost, *Calculus of variations*, Cambridge Univ. Press, Cambridge, 1998.
  - [20] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, 204, Elsevier, Amsterdam, 2006.
  - [21] M. Klimek, *On solutions of linear fractional differential equations of a variational type*. The Publishing Office of Czenstochowa University of Technology, Czestochowa, 2009.
  - [22] Y. Kosmann-Schwarzbach, *The Noether theorems: Invariance and conservation laws in the twentieth century*.

- Springer, New York, 2011.
- [23] C. F. Lorenzo, T. T. Hartley, *Variable order and distributed order fractional operators*. Nonlinear Dynam. **29** (2002), 57–98.
  - [24] A. B. Malinowska, D. F. M. Torres, *Introduction to the fractional calculus of variations*. Imp. Coll. Press, London, 2012.
  - [25] D. Mozyrska, D. F. M. Torres, *Minimal modified energy control for fractional linear control systems with the Caputo derivative*. Carpathian J. Math. **26** (2010), 210–221. [arXiv:1004.3113](#)
  - [26] D. Mozyrska, D. F. M. Torres, *Modified optimal energy and initial memory of fractional continuous-time linear systems*. Signal Process. **91** (2011), 379–385. [arXiv:1007.3946](#)
  - [27] T. Odziejewicz, A. B. Malinowska, D. F. M. Torres, *Variable order fractional variational calculus for double integrals*. Proc. 51st IEEE Conference on Decision and Control, Dec. 10-13, 2012, Maui, Hawaii, art. no. 6426489, 6873–6878. [arXiv:1209.1345](#)
  - [28] T. Odziejewicz, A. B. Malinowska, D. F. M. Torres, *Fractional variational calculus of variable order*. Advances in Harmonic Analysis and Operator Theory (Eds: A. Almeida, L. P. Castro, F.-O. Speck), The Stefan Samko Anniversary Volume, Birkhäuser, Basel, 2013, 291–301. [arXiv:1110.4141](#)
  - [29] M. D. Ortigueira, *Fractional calculus for scientists and engineers*. Lecture Notes in Electrical Engineering, 84, Springer, Dordrecht, 2011.
  - [30] I. Podlubny, *Fractional differential equations*. Mathematics in Science and Engineering, 198, Academic Press, San Diego, CA, 1999.
  - [31] L. E. S. Ramirez, C. F. M. Coimbra, *On the selection and meaning of variable order operators for dynamic modeling*. Int. J. Differ. Equ. **2010** (2010), Art. ID 846107, 16 pp.
  - [32] L. E. S. Ramirez, C. F. M. Coimbra, *On the variable order dynamics of the nonlinear wake caused by a sedimenting particle*. Phys. D **240** (2011), 1111–1118.
  - [33] F. Riewe, *Nonconservative Lagrangian and Hamiltonian mechanics*. Phys. Rev. E (3) **53** (1996), 1890–1899.
  - [34] F. Riewe, *Mechanics with fractional derivatives*. Phys. Rev. E (3) **55** (1997), 3581–3592.
  - [35] B. Ross, S. G. Samko, *Fractional integration operator of a variable order in the Holder spaces  $H^{\lambda(x)}$* . Internat. J. Math. Math. Sci. **18** (1995), 777–788.
  - [36] S. G. Samko, *Fractional integration and differentiation of variable order*. Anal. Math. **21** (1995), 213–236.
  - [37] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives*, translated from the 1987 Russian original, Gordon and Breach, Yverdon, 1993.
  - [38] S. G. Samko, B. Ross, *Integration and differentiation to a variable fractional order*. Integral Transform. Spec. Funct. **1** (1993), 277–300.
  - [39] J. A. Tenreiro Machado, V. Kiryakova, F. Mainardi, *Recent history of fractional calculus*. Commun. Nonlinear Sci. Numer. Simul. **16** (2011), 1140–1153.
  - [40] B. van Brunt, *The calculus of variations*. Universitext, Springer, New York, 2004.